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Decomposing trees with large diameter

Romain Ravaux *

Versailles University, Prism laboratory, 78035 Versailles cedex, France

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ABSTRACT

An n-vertex graph is said to be decomposable for a partition $(\lambda_1, \ldots, \lambda_p)$ of the integer n if there exists a sequence $(V_1, \ldots, V_p)$ of connected vertex-disjoint subgraphs with $|V_i| = \lambda_i$. An n-vertex graph is said to be decomposable if this graph is decomposable for all the partitions of the integer n. We are interested in decomposable trees with large diameter. We show that any n-vertex tree T with diameter $n-\alpha$ is decomposable for all the partitions of n which contain at least α distinct integers. This structural result provides an algorithm to decide if an n-vertex tree T with diameter $n-\alpha$ is decomposable in time $n^{O(\alpha)}$.

1. Introduction

We call a partition of the integer n a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ such that for any $i \in \{1, \ldots, p\}$, $\lambda_i \in \mathbb{N}^*$, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ and $\sum_{i=1}^{p} \lambda_i = n$. Consider an n-vertex graph $G = (V, E)$ and let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a partition of n. A decomposition of G for λ is a partition $V_1, \ldots, V_p$ of V such that for all $1 \leq i \leq p$, we have $|V_i| = \lambda_i$, and the subgraph of G induced by any subset V_i is connected. Such a partition $V_1, \ldots, V_p$ of V is called a (G, λ)-partition. The graph G is said to be decomposable if and only if for all partition λ of n the graph G is decomposable for λ.

Some famous results concern k-connected graphs. Respectively in 1976 and 1977, Györi [1] and Lovász [2] have shown that any n-vertex k-connected graph G is decomposable for all partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n which contain k parts. However their proofs do not yield any polynomial-time algorithms. Polynomial-time algorithms have been given for the cases $k = 2$ and $k = 3$ [3,4]. Another has been given for the case $k = 4$ restricted to the planar graphs [5].

Various properties on decomposable trees have been shown during these last years. It has been shown [6] that decomposable trees are of maximum degree at most 6. Later [7] this bound has been decreased to 4 . Conditions on decomposable star-like trees (decomposable trees containing exactly one vertex of degree greater than 2) have been exhibited in [8]. A characterisation of homeomorphism classes containing decomposable trees with an arbitrarily large minimal distance between all pairs of distinct vertices of degree different from 2 has been given in [9]. These homeomorphism classes are exactly the set of combs.

From an algorithmic point of view, the polynomial time decidability of decomposable trees is still open [7]. It has been shown [10] that deciding whether a given tripode (three disjoint chains connected by one extremity) is decomposable can be done by a polynomial algorithm. Their algorithm studies a subset of the set of partitions of n. This subset has a polynomial size, whereas the set of partitions of n has a size $\Omega(e^{\sqrt{n}})$. Moreover for each partition λ of their subset, deciding if the tripode is λ decomposable can be done in polynomial time. We know that the n-vertex path is decomposable, thus we don't have to test any partition of n. Now if we consider a tree which contains a very long a path, we can assume that we will just have to test a small subset of all partitions of n too.

* Tel.: +33 130646933.
E-mail address: rravaux@wanadoo.fr.

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In this paper we focus on trees with a large diameter. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ be a partition of the integer n, the spectrum of λ is defined by $sp(\lambda) = \bigcup_{i=1}^{p}\{\lambda_i\}$. We show that any n-vertex tree T with diameter $n - \alpha$ is λ-decomposable for all partitions $\lambda = (\lambda_1, \ldots, \lambda_p)$ of n with $|sp(\lambda)| \geq \alpha$. This structural result provides an algorithm to decide if an n-vertex tree T with diameter $n - \alpha$ is decomposable in time $n^{O(\alpha)}$.

2. Integers' permutation whose partial sums avoid a given set of values

In this section we show that any n-vertex tree T with diameter $n - \alpha$ is decomposable for all the partitions of n with a spectrum cardinality greater or equal to α (see Proposition 1). The proof relies on the Lemma below.

Lemma 1. *Let $I \subseteq \{1, \ldots, n-1\}$ be such that $|I| = \alpha - 1$. For all partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of n such that $|sp(\lambda)| \geq \alpha$, there exists a permutation $\pi = (\pi_1, \ldots, \pi_p)$ of λ such that for all $1 \leq i \leq p-1$ we have $\sum_{j=1}^{i}\pi_j \notin I$.*

Proof. We consider a set I of positive integers such that $I \subseteq \{1, \ldots, n-1\}$ and $|I| = \alpha - 1$. For all partition λ of n such that $|sp(\lambda)| \geq \alpha$ we have to show that there exists a permutation π of λ whose partial sums are not all in I. Then we call I the set of forbidden integers. Let $P = \{1, \ldots, n\} \setminus I$. We call P the set of possible integers. Notice that a partial sum which is not in I, is in P.

We proceed by induction on $|I|$. It is trivial for the base case $|I| = 0$.

Now suppose that Lemma 1 holds for $|I| \leq \alpha - 1$, we are going to show that Lemma 1 is true for $|I| = \alpha$. Consider a partition λ such that $|sp(\lambda)| \geq \alpha + 1$. Then there exist $s_1, s_2, \ldots, s_{\alpha+1}$ such that $s_1 < s_2 < \cdots < s_{\alpha+1}$ and for any $1 \leq i \leq \alpha + 1$ we have $s_i \in sp(\lambda)$ with $s_{\alpha+1} = \lambda_1$. Recall that λ_1 is the greatest part of the partition λ since $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ by hypothesis. For any $1 \leq i \leq \alpha + 1$, we denote $S_i = \sum_{j=1}^{i} s_j$, with $S_0 = 0$. We denote $r_1 \leq r_2 \leq \cdots \leq r_{p-(\alpha+1)}$ the $p - (\alpha + 1)$ other parts of λ. For any $1 \leq i \leq p - (\alpha + 1)$, we denote $R_i = \sum_{j=1}^{i} r_j$, with $R_0 = 0$ and $R = R_{p-(\alpha+1)}$. At the rank $\alpha + 1$ we have $|I| = \alpha$. We denote $I = \{\theta_1, \ldots, \theta_\alpha\}$ with $\theta_1 < \cdots < \theta_\alpha$. For all sub-partition $(\lambda_{i_1}, \ldots, \lambda_{i_z})$ of λ, we will denote $\lambda - (\lambda_{i_1}, \ldots, \lambda_{i_z})$ the partition of $n - \sum_{j=1}^{z} \lambda_{i_j}$ obtained from λ by removing $\lambda_{i_1}, \ldots, \lambda_{i_z}$.

Case 1: $R + s_{\alpha+1} > \theta_1$. In this case we search $i \in \{0, \ldots, p - (\alpha + 1)\}$ such that $R_i < \theta_1$ and $R_i + s_{\alpha+1} > \theta_1$.

Case 1.a: There exists a such i, with $R_i + s_{\alpha+1} \in P$. In this case we can choose $\pi_1 = r_1, \ldots, \pi_i = r_i, \pi_{i+1} = s_{\alpha+1}$. We will say that "it remains to choose $\pi_{i+2}, \ldots, \pi_p$ on the interval $]R_i + s_{\alpha+1}, n]$", because for any ℓ in $\{i+2, \ldots, \alpha+1\}$ we have $\sum_{j=1}^{\ell}\pi_j > R_i + s_{\alpha+1}$. We have $|]R_i + s_{\alpha+1}, n] \cap I| \leq \alpha - 1$ and $|sp(\lambda - (r_1, \ldots, r_i, s_{\alpha+1}))| \geq \alpha$. By Hypothesis of induction, on the interval $]R_i + s_{\alpha+1}, n]$, there exists a permutation $\pi_{i+2}, \ldots, \pi_p$ of $\lambda - (r_1, \ldots, r_i, s_{\alpha+1})$ such that for any ℓ in $\{i+2, \ldots, p\}$ we have $\sum_{j=1}^{\ell}\pi_j \in P$.

Case 1.b: There exists a such i, with $R_i + s_{\alpha+1} \in I$. Then we have $R_i + s_{\alpha+1} = \theta_u$ with $u \geq 2$.
In this case there exists $t \in \{1, \ldots, \alpha\}$ satisfying $R_i + s_t \in P$ and $R_i + s_t + s_{\alpha+1} \in P$. Indeed, for each $\ell \in \{1, \ldots, \alpha\}$ we have $R_i + s_\ell < \theta_u$ and $R_i + s_\ell + s_{\alpha+1} > \theta_u$. And if for each $\ell \in \{1, \ldots, \alpha\}$ we had $R_i + s_\ell \in I$ or $R_i + s_\ell + s_{\alpha+1} \in I$, then we would have $|([1, \theta_u[\cup]\theta_u, n]) \cap I| \geq \alpha$ and thus $|[1, n] \cap I| \geq \alpha + 1$. This is impossible because we have $|I| = \alpha$. Thus we can choose $\pi_1 = r_1, \ldots, \pi_i = r_i, \pi_{i+1} = s_t, \pi_{i+2} = s_{\alpha+1}$. It remains to choose $\pi_{i+3}, \ldots, \pi_p$ on the interval $]R_i + s_t + s_{\alpha+1}, n]$. We have $|]R_i + s_t + s_{\alpha+1}, n] \cap I| \leq \alpha - u$ and thus $|]R_i + s_t + s_{\alpha+1}, n] \cap I| \leq \alpha - 2$, and we have $|sp(\lambda - (r_1, \ldots, r_i, s_t, s_{\alpha+1}))| \geq \alpha - 1$. By induction hypothesis, on the interval $]R_i + s_t + s_{\alpha+1}, n]$, there exists a permutation $\pi_{i+3}, \ldots, \pi_p$ of $\lambda - (r_1, \ldots, r_i, s_t, s_{\alpha+1})$ such that for any ℓ in $\{i+3, \ldots, p\}$ we have $\sum_{j=1}^{\ell}\pi_j \in P$.

Case 1.c: There doesn't exist $i \in \{0, \ldots, p - (\alpha + 1)\}$ satisfying $R_i < \theta_1$ and $R_i + s_{\alpha+1} > \theta_1$. In this case, there exists $i \in \{0, \ldots, p - (\alpha + 1) - 1\}$ such that $R_i < \theta_1$ and $R_i + s_{\alpha+1} = \theta_1$ with $r_{i+1} = s_{\alpha+1}$. We can make the same reasoning as in the previous case (case 1.b) taking $u = 1$ and replacing $s_{\alpha+1}$ by r_{i+1} because we have $r_{i+1} = s_{\alpha+1}$. We obtain the following result: there exists $t \in \{1, \ldots, \alpha\}$ such that $R_i + s_t \in P$ and $R_i + s_t + r_{i+1} \in P$. We have $|]R_i + s_t + r_{i+1}, n] \cap I| \leq \alpha - 1$ and $|sp(\lambda - (r_1, \ldots, r_i, r_{i+1}, s_t))| \geq \alpha$. By induction hypothesis, on the interval $]R_i + s_t + r_{i+1}, n]$, there exists a permutation $\pi_{i+3}, \ldots, \pi_p$ of $\lambda - \{r_1, \ldots, r_i, r_{i+1}, s_t\}$ such that for any ℓ in $\{i+3, \ldots, \alpha+1\}$ we have $\sum_{j=1}^{\ell}\pi_j \in P$.

Case 2: $R + s_{\alpha+1} \leq \theta_1$. In this case we have $R + s_1 < \theta_1$ and we search the smallest i in $\{2, \ldots, \alpha\}$ satisfying $R + S_{i-1} < \theta_{i-1}$ and $R + S_i \geq \theta_i$.

Case 2.a: There doesn't exist such a i. We need to give a claim. Let $I = \{\theta_1, \ldots, \theta_t\}$ be a set of positive integers such that $I \subseteq \{1, \ldots, n-1\}$ with $\theta_1 < \cdots < \theta_t$. We denote $\bar{I} = \{\bar{\theta}_1, \ldots, \bar{\theta}_t\}$ the set of positive integers obtained from I and n by taking for any $i \in \{1, \ldots, t\}$, $\bar{\theta}_i = n - \theta_{t+1-i}$. We have $\bar{I} \subseteq \{1, \ldots, n-1\}$ and $|\bar{I}| = t$ with $\bar{\theta}_1 < \cdots < \bar{\theta}_t$. Notice that $\bar{\bar{I}} = I$. Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a partition of n. Let $\pi = (\pi_1, \ldots, \pi_p)$ be a permutation of λ. We denote $\bar{\pi} = (\bar{\pi}_1, \ldots, \bar{\pi}_p)$ the permutation of λ obtained from π by taking for any $i \in \{1, \ldots, p\}$, $\bar{\pi}_i = \pi_{p+1-i}$. Notice that $\bar{\bar{\pi}} = \pi$.

Claim 1. *Let $I = \{\theta_1, \ldots, \theta_t\}$ be a set of natural integers such that $I \subseteq \{1, \ldots, n-1\}$ with $\theta_1 < \cdots < \theta_t$. Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a partition of n. Let π be a permutation of λ.*
The following two facts are equivalent:
(a) *For all $i \in \{1, \ldots, p\}$, $\sum_{j=1}^{i}\pi_j \notin I$.*
(b) *For all $i \in \{1, \ldots, p\}$, $\sum_{j=1}^{i}\bar{\pi}_j \notin \bar{I}$.*

Remember that we are in the case where no $i \in \{2, \ldots, \alpha\}$ satisfies both $R + S_{i-1} < \theta_{i-1}$ and $R + S_i \geq \theta_i$. Then we have $R + S_\alpha < \theta_\alpha$, we obtain $n - (R + S_\alpha) > n - \theta_\alpha$, and thus $s_{\alpha+1} > n - \theta_\alpha$. We have $n - \theta_\alpha = \bar{\theta}_1$. We are in the case where we have $s_{\alpha+1} > \bar{\theta}_1$ and thus $R + s_{\alpha+1} > \bar{\theta}_1$. From Case 1 previously studied, we can affirm that there exists a permutation $\bar{\pi}$ of λ such that for any $i \in \{1, \ldots, p\}$ we have $\sum_{j=1}^{i} \bar{\pi}_j \notin \bar{I}$. From Claim 1 we can affirm that there exists a permutation π of λ such that for any $i \in \{1, \ldots, p\}$ we have $\sum_{j=1}^{i} \pi_j \notin I$.

Case 2.b: There exists $i \in \{2, \ldots, \alpha\}$ such that $R + S_{i-1} < \theta_{i-1}$ and $R + S_i \geq \theta_i$, with $R + S_{i-1} \in P$.

We have $|[1, R + S_{i-1}] \cap I| < i - 1$ and $|sp((r_1, \ldots, r_{p-(\alpha+1)}, s_1, \ldots, s_{i-1}))| \geq i - 1$. By induction hypothesis, on the interval $[1, R + S_{i-1}]$, there exists a permutation $\pi_1, \ldots, \pi_{p-(\alpha+1)+(i-1)}$ of $(r_1, \ldots, r_{p-(\alpha+1)}, s_1, \ldots, s_{i-1})$ such that for any $\ell \in \{1, \ldots, p - (\alpha + 1) + (i - 1)\}$ we have $\sum_{j=1}^{\ell} \pi_j \in P$. Remember that $R + S_i \geq \theta_i$. Thus for any $\ell \in \{i + 1, \ldots, \alpha + 1\}$ we have $R + S_i - s_i + s_\ell > \theta_i$ because $s_\ell > s_i$, and thus $R + S_{i-1} + s_\ell > \theta_i$. We have also $|]\theta_i, n] \cap I| = \alpha - i$ and $|\{i + 1, \ldots, \alpha + 1\}| = \alpha + 1 - i$. Thus there exists $t \in \{i + 1, \ldots, \alpha + 1\}$ such that $R + S_{i-1} + s_t \in P$. Thus we can choose $\pi_{p-(\alpha+1)+(i-1)+1} = s_t$. We have $|]R + S_{i-1} + s_t, n] \cap I| \leq \alpha - i$ because $R + S_{i-1} + s_t > \theta_i$. And we have $|sp((s_i, \ldots, s_{\alpha+1}) - (s_t))| = \alpha + 1 - i$. By induction hypothesis, on the interval $]R + S_{i-1} + s_t, n]$, there exists a permutation $\pi_{p-(\alpha+1)+(i-1)+2}, \ldots, \pi_p$ of $(s_i, \ldots, s_{\alpha+1}) - (s_t)$ such that for any $\ell \in \{p - (\alpha + 1) + (i - 1) + 2, \ldots, p\}$ we have $\sum_{j=1}^{\ell} \pi_j \in P$.

Case 2.c: There exists $i \in \{2, \ldots, \alpha\}$ such that $R + S_{i-1} < \theta_{i-1}$ and $R + S_i \geq \theta_i$, with $R + S_{i-1} \in I$. We have $R + S_{i-1} = \theta_u$ with $u \leq i - 2$. We can again distinguish two cases.

Case 2.c.1: There exists $t \in \{1, \ldots, i - 1\}$ such that $R + S_{i-1} - s_t \in P$ and $R + S_{i-1} - s_t + s_i \in P$. In this case we have $|[1, R + S_{i-1} - s_t] \cap I| < i - 2$ because $R + S_{i-1} - s_t < R + S_{i-1}$. And we have $|sp((r_1, \ldots, r_{p-(\alpha+1)}, s_1, \ldots, s_{i-1}) - (s_t))| \geq i - 2$. Thus by induction hypothesis, on the interval $[1, R + S_{i-1} - s_t]$, there exists a permutation $\pi_1, \ldots, \pi_{p-(\alpha+1)+(i-1)-1}$ of $(r_1, \ldots, r_{p-(\alpha+1)}, s_1, \ldots, s_{i-1}) - (s_t)$ such that for any ℓ in $\{1, \ldots, p - (\alpha + 1) + (i - 1) - 1\}$ we have $\sum_{j=1}^{\ell} \pi_j \in P$.
Since $R + S_{i-1} - s_t + s_i \in P$ we can choose $\pi_{p-(\alpha+1)+(i-1)} = s_i$. For any $\ell \in \{i + 1, \ldots, \alpha + 1\}$ we have $R + S_i - s_t + s_\ell > R + S_i$ and thus $R + S_i - s_t + s_\ell > \theta_i$ because $R + S_i \geq \theta_i$.

We have also $|]R + S_i, n] \cap I| \leq \alpha - i$ and $|\{i + 1, \ldots, \alpha + 1\}| = \alpha + 1 - i$, thus there exists $z \in \{i + 1, \ldots, \alpha + 1\}$ such that $R + S_i - s_t + s_z \in P$. Hence we can choose $\pi_{p-(\alpha+1)+(i-1)+1} = s_z$.

We have $|]R + S_i - s_t + s_z, n] \cap I| \leq \alpha - i$. We have $|sp((s_{i+1}, \ldots, s_{\alpha+1}) - (s_z) + (s_t))| = \alpha + 1 - i$. By induction hypothesis, on the interval $]R + S_i - s_t + s_z, n]$, there exists a permutation $\pi_{p-(\alpha+1)+(i-1)+2}, \ldots, \pi_p$ of $(s_{i+1}, \ldots, s_{\alpha+1}) - (s_z) + (s_t)$ such that for any ℓ in $\{p - (\alpha + 1) + (i - 1) + 2, \ldots, p\}$ we have $\sum_{j=1}^{\ell} \pi_j \in P$.

Case 2.c.2: There doesn't exist $t \in \{1, \ldots, i - 1\}$ such that $R + S_{i-1} - s_t \in P$ and $R + S_{i-1} - s_t + s_i \in P$. In this case, for any $t \in \{1, \ldots, i-1\}$ we have $R + S_{i-1} - s_t \in I$ or $R + S_{i-1} - s_t + s_i \in I$. We denote $t_{min} = min(\{t \in \{1, \ldots, i-1\} : R + S_{i-1} - s_t \in P\})$. Notice that there exists $t \in \{1, \ldots, i - 1\}$ such that $R + S_{i-1} - s_t \in P$, because for any $t \in \{1, \ldots, i - 1\}$ we have $R + S_{i-1} - s_t < \theta_u$ with $u \leq i - 2$. We have $|[1, R + S_{i-1} - s_{t_{min}}] \cap I| < i - 2$ because remember that we have $R + S_{i-1} = \theta_u$ with $u \leq i - 2$. We have $|sp((r_1, \ldots, r_{p-(\alpha+1)}, s_1, \ldots, s_{i-1}) - (s_{t_{min}}))| \geq i - 2$. By induction hypothesis, on the interval $[1, R + S_{i-1} - s_{t_{min}}]$, there exists a permutation $\pi_1, \ldots, \pi_{p-(\alpha+1)+(i-1)-1}$ of $(r_1, \ldots, r_{p-(\alpha+1)}, s_1, \ldots, s_{i-1}) - (s_{t_{min}})$ such that for any ℓ in $\{1, \ldots, p - (\alpha + 1) + (i - 1) - 1\}$ we have $\sum_{j=1}^{\ell} \pi_j \in P$.

For any $t \in \{1, \ldots, i - 1\}$ we have either $R + S_{i-1} - s_t \in I$ with $R + S_{i-1} - s_t \in [1, \theta_u[$, or $R + S_{i-1} - s_t \in P$ and thus $R + S_{i-1} - s_t + s_i \in I$ with $R + S_{i-1} - s_t + s_i \in]\theta_u, R + S_{i-1} - s_{t_{min}} + s_i]$. Thus we have $|([1, \theta_u[\cup]\theta_u, R + S_{i-1} - s_{t_{min}} + s_i]) \cap I| \geq i - 1$. It follows that $|[1, R + S_{i-1} - s_{t_{min}} + s_i] \cap I| \geq i$. For any $\ell \in \{i + 1, \ldots, \alpha + 1\}$ we have $R + S_{i-1} - s_{t_{min}} + s_\ell > R + S_{i-1} - s_{t_{min}} + s_i$. We have also $|]R + S_{i-1} - s_{t_{min}} + s_i, n] \cap I| \leq \alpha - i$ and $|\{i + 1, \ldots, \alpha + 1\}| = \alpha + 1 - i$, thus there exists $t \in \{i + 1, \ldots, \alpha + 1\}$ such that $R + S_{i-1} - s_{t_{min}} + s_t \in P$. Hence we can choose $\pi_{p-(\alpha+1)+(i-1)} = s_t$. We have $|]R + S_{i-1} - s_{t_{min}} + s_t, n] \cap I| \leq \alpha - i$ and $|sp((s_i, \ldots, s_{\alpha+1}) - (s_t) + (s_{t_{min}}))| = \alpha + 2 - i$. By induction hypothesis, on the interval $]R + S_{i-1} - s_{t_{min}} + s_t, n]$, there exists a permutation $\pi_{p-(\alpha+1)+(i-1)+1}, \ldots, \pi_p$ of $(s_i, \ldots, s_{\alpha+1}) - (s_t) + (s_{t_{min}})$ such that for any ℓ in $\{p - (\alpha + 1) + (i - 1) + 1, \ldots, p\}$ we have $\sum_{j=1}^{\ell} \pi_j \in P$.

We have shown that Lemma 1 is true at rank $\alpha + 1$. □

Proposition 1. *Consider an n-vertex tree $T = (V, E)$ with diameter $n - \alpha$. The tree T is decomposable for all partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of n with $|sp(\lambda)| \geq \alpha$.*

Proof. We first find a path $(x_0, x_1, \ldots, x_{n-\alpha})$ in time $n^{O(1)}$. The graph $F = (V_F, E_F)$ such that $V_F = V$ and $E_F = E - \{\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-\alpha-1}, x_{n-\alpha}\}\}$ is a forest composed of $n - \alpha + 1$ trees, each one containing one vertex of the chain $(x_0, x_1, \ldots, x_{n-\alpha})$. For all $0 \leq i \leq n - \alpha$, we denote A_i the set of vertices of the connected component of (V_F, E_F) containing the vertex x_i. We shall prove that for every partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of n with $|sp(\lambda)| \geq \alpha$, there exists a (T,λ)-partition $V_1, V_2, \ldots, V_p$ of $V(G)$ with the following property: for all $0 \leq i \leq n - \alpha$, there exists $j \in \{1, \ldots, p\}$ such that $A_i \subseteq V_j$. Then, for all $0 \leq i \leq n - \alpha$ the vertices of the tree induced by A_i are included in one of the sets of the (T,λ)-partition $V_1, V_2, \ldots, V_p$. Such a (T,λ)-partition is called a (T,λ)-clean partition (see Fig. 1).

Consider the set of positive integers $P = \bigcup_{i=0}^{n-\alpha} \{\sum_{j=0}^{i} |A_j|\}$ and the set of natural integers $I = \{1, \ldots, n\} \setminus P$. We call P the set of *possible integers* and I the set of *forbidden integers*. By definition, P and I form a partition of $\{1, \ldots, n\}$, with $n \in P$, $|P| = n - \alpha + 1$ and $|I| = \alpha - 1$. By Lemma 1, there exists a permutation $\pi_1, \ldots, \pi_p$ of λ such that for all $1 \leq i \leq p$ we have $\sum_{j=1}^{i} \pi_j \in P$. This permutation yields a (T,λ)-clean partition of V. □

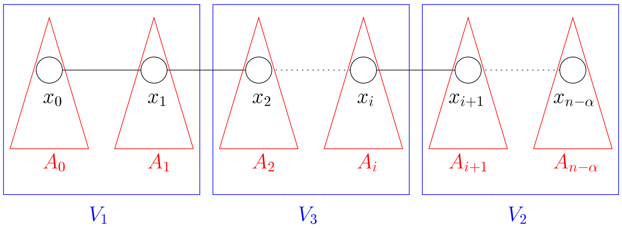

Fig. 1. A tree T with diameter $n-\alpha$ and a (T,p)-clean partition $\{V_1, V_2, V_3\}$.

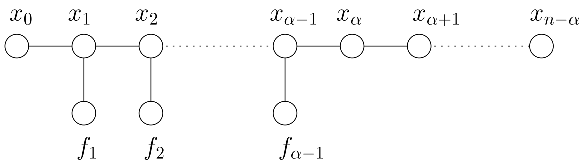

Fig. 2. A tree T with diameter $n-\alpha$ witnessing the tightness of Proposition 1.

Notice that Proposition 1 is tight. Indeed, consider an integer α. Let $n = \sum_{i=1}^{\alpha-1} 2i$. Consider the partition $\lambda = (2, 4, 6, \dots, 2\alpha-2)$ of n. We have $|sp(\lambda)| = \alpha - 1$. The n-vertex tree $T = (V, E)$ with diameter $n-\alpha$ in Fig. 2 is not decomposable for the partition λ, since for all $i \in \{1, \dots, \alpha-1\}$, if the vertex x_0 belongs to V_i of size $2i$, then one of the vertices $\{f_1, \dots, f_i\}$ will be isolated and there is no part of size 1 in λ.

3. Algorithmic results

We know by Proposition 1 that any tree $T = (V, E)$ with diameter $n-\alpha$ is decomposable for all partitions λ with $|sp(\lambda)| \geq \alpha$. To decide if such a tree T is decomposable, one must check whether it is decomposable by all other partitions. In this section, we present an algorithm which does so in time $n^{O(\alpha)}$. We first consider the case of a single partition.

Proposition 2. *Consider an n-vertex tree $T = (V, E)$ with diameter $n-\alpha$. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ be a partition of n. Deciding if the tree T is decomposable for the partition λ can be done in time $n^{O(\alpha)}$.*

Proof. If $|sp(\lambda)| \geq \alpha$, then by Proposition 1, we know that T is decomposable for λ. Suppose now that $|sp(\lambda)| \leq \alpha - 1$. Let $(x_0, x_1, \dots, x_{n-\alpha})$ be an elementary path of length $n-\alpha$. Such a path can be computed in polynomial time. Let $\{u_1, u_2, \dots, u_{\alpha-1}\} = V \setminus \{x_0, x_1, \dots, x_{n-\alpha}\}$. We intend to verify if there exists a (T,λ)-partition. The first step of the process consists in generating all the systems $\{S_1, \dots, S_m\}$ of pairwise distinct subsets of V with $m \leq \alpha - 1$, such that for any $i \in \{1, \dots, m\}, S_i \cap \{u_1, u_2, \dots, u_{\alpha-1}\} \neq \emptyset$ and the subgraph of T induced by S_i is connected, and $\{u_1, u_2, \dots, u_{\alpha-1}\} \subseteq \cup_{i=1}^m S_i$.

We first have to choose the number of parts m ($\alpha - 1$ possibilities). Next we have to choose a sub-partition $(c_1, \dots, c_m)$ of λ (for each c_i we have at most $\alpha - 1$ possibilities because we have $|sp(\lambda)| \leq \alpha - 1$, that gives at most $(\alpha-1)^{\alpha-1}$ possible sub-partitions). The sub-partition $(c_1, \dots, c_m)$ of n represents the chosen sizes of parts. Next we generate all the surjective mappings f from $\{1, \dots, \alpha-1\}$ onto $\{1, \dots, m\}$: $f(i) = j$ means that the vertex u_i belongs to S_j. (For a chosen m and chosen sizes of part, there are at most $m^{\alpha-1}$ possibilities with $m \leq \alpha - 1$, and then there are at most $(\alpha-1)^{\alpha-1}$ possibilities). Now we have to place for any $i \in \{1, \dots, m\}$ the part S_i on the tree T. Notice that it is not always possible to place the parts S_i.

We are going to use the notation A_j defined in the beginning of the proof of Proposition 1. For any $i \in \{1, \dots, m\}$, consider

$$g(i) = \min\{j \in \{0, \dots, n-\alpha\} : A_j \cap S_i \neq \emptyset\}.$$

Let $d(i) = \max\{j \in \{0, \dots, n-\alpha\} : A_j \cap S_i \neq \emptyset\}$. Adding the vertices $x_{g(i)}, x_{g(i)+1}, \dots, x_{d(i)}$ in the subset S_i is a necessary condition (but not sufficient) for the subgraph of T induced by S_i to be connected. If c_i is large enough to add these vertices in S_i, then we have to verify if the subgraph of T induced by S_i is connected. This can be done in time $O(n)$. If the (verification) result is positive, we have to find a set $\bar{S_i} \supseteq S_i$ with $|\bar{S_i}| = c_i$ such that the subgraph of T induced by $\bar{S_i}$ is connected (using some additional vertices of paths $(x_0, \dots, x_{g(i)})$ and $(x_{d(i)}, \dots, x_{n-\alpha})$). For any $i \in \{1, \dots, m\}$, there are at most $O(n)$ possibilities to complete the set S_i with these vertices.

Thus the number of such possible tuples $S_1, \dots, S_m$ is $O(\alpha^{2\alpha} n^\alpha)$.

For each possible tuple $S_1, \dots, S_m$, we have to remove the vertices which belong to $\cup_{i=1}^m S_i$. At this stage, it remains a forest F which is composed of at most α connected components, each one being a chain. It remains a sub-partition p' of p, which contains integers of p which have not be used in the previous step. Deciding if F is decomposable for the partition p' is equivalent to solving an α-subset sum problem, with $\alpha \leq n$ and n coded in unary. By dynamic programming, we can decide if F is decomposable for the partition p' in time $n^{O(\alpha)}$. Thus we can decide if T is decomposable for the partition λ in time $n^{O(\alpha)}$. $\square$

Theorem 1. *Consider an n-vertex tree $T = (V, E)$ with diameter $n-\alpha$. Deciding if the tree T is decomposable can be done in time $n^{O(\alpha)}$.*

Proof. By Proposition 1, we know that the tree T is decomposable for all partitions λ of n with $|sp(\lambda)| \geq \alpha$. Thus it remains to study the partitions λ of n such that $|sp(\lambda)| < \alpha$. Consider an integer n and an integer α. The number of partitions λ of n with $|sp(\lambda)| \leq \alpha - 1$ is $O(\alpha n^{2\alpha})$ and we can generate them in time $O(\alpha n^{2\alpha+1})$. By Proposition 2, deciding if the tree T is decomposable can be done in time $n^{O(\alpha)}$. $\square$

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